Power-law distribution of degree–degree distance: A better representation of the scale-free property of complex networks

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Whether real-world complex networks are scale free or not has long been controversial. Recently, in Broido and Clauset [A. D. Broido, A. Clauset, *Nat. Commun.* 10, 1017 (2019)], it was claimed that the degree distributions of real-world networks are rarely power law under statistical tests. Here, we attempt to address this issue by defining a fundamental property possessed by each link, the degree–degree distance, the distribution of which also shows signs of being power law by our empirical study. Surprisingly, although full-range statistical tests show that degree distributions are not often power law in real-world networks, we find that in more than half of the cases the degree–degree distance distributions can still be described by power laws. To explain these findings, we introduce a bidirectional preferential selection model where the link configuration is a randomly weighted, two-way selection process. The model does not always produce solid power-law distributions but predicts that the degree–degree distance distribution exhibits stronger power-law behavior than the degree distribution of a finite-size network, especially when the network is dense. We test the strength of our model and its predictive power by examining how real-world networks evolve into an overly dense stage and how the corresponding distributions change. We propose that being scale free is a property of a complex network that should be determined by its underlying mechanism (e.g., preferential attachment) rather than by apparent distribution statistics of finite size. We thus conclude that the degree–degree distance distribution better represents the scale-free property of a complex network.

complex network | scale-free property | power-law distribution | degree–degree distance | bidirectional preferential selection

The study of scale-free complex networks has undergone an exponentially rapid and highly controversial development in recent decades (1–7). Since its first appearance in ref. 1, the concept of scale-free complex networks has broadened itself more rapidly than anyone expected, and its abundance in real life now arguably encompasses many areas from fundamental physics (8–11) to social systems (12, 13). Consequently, scale-free complex networks are now commonly regarded as an essential substrate for studying many other facets in network science (14–23), such as percolation (24–26), epidemic spreading (27–29), and information diffusion (30–33).

Unfortunately, for all of its wide influence, the most basic definition of a network being "scale free" has never reached a common sense agreement (34, 35). Across the broad literature, the definition may originate in the statement that the statistics of the degree distribution $P(k)$ follows a precise or imprecise power law. It may also arise from the acknowledgment that being scale free is an intrinsic network property determined by some family of mechanisms of network generation (e.g., the preferential attachment mechanism) (36, 37). The ambiguity herein depreciates pertinent studies and deepens the controversy.

The degree distribution-based definition implies an equivalence between scale free and "power law." In other words, being scale free is treated as an explicit behavior, since for any $P(k) \propto k^{-\alpha}$, one has $P((1+\epsilon)k) \simeq (1+\epsilon)^{-\alpha} P(k)$ where ϵ is an infinitesimal transformation of the scale (i.e., dilation). Many studies have, however, challenged on statistical grounds that the degree distributions are not rigorously power law in real-world complex networks (38–46). Instead, they follow alternative, nonpower-law distributions, which are statistically preferred. Thus, the abundance of scale-free networks seems to be a negative conclusion (34). Nevertheless, such a statistics-based argument is imperfect. First, even for synthetic scale-free networks, the analysis in ref. 34 may not give the strongest statistical significance of power laws. Second, one cannot fully eliminate the possibility that some nonpower-law distributions are merely due to statistical limitations by finite size or binning (35). For example, theoretically a fluctuation-induced exponential cutoff may exist in the extreme upper tail of any power-law distribution.

We here argue that the equivalence between scale free and power law is questionable. We introduce a fundamental property, the degree–degree distance, defined for each link of a network. Full-range statistical tests show that, although degree distributions are not often power law, degree–degree distance distributions can still be described by power laws in many real-world complex networks. Hence, the distribution-based

Significance

It is hard to decide if a real-world complex network is scale free, as it has never reached an agreement on how to confirm whether the network exhibits statistically favored power-law distributions. Here, we define a fundamental property—the degree–degree distance—possessed by each link, the distribution of which usually exhibits a stronger power law than the degree distribution of a finite-size network as indicated by empirical and statistical studies. A bidirectional preferential selection model is introduced to explain and reproduce this finding, which implies that being scale free as a property should not be defined by apparent statistics but determined by the underlying mechanism. We conclude that power-law degree–degree distance distribution better represents the scale-free property.

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definition of being scale free is incomplete: different distributions may contradict each other in the appearance of their power-law statistics. Our network generation model also confirms our findings via analytical solutions and finite-size simulation. We are convinced that the scale-free property of a network is an implicit property that should not be defined by apparent statistics on finite-size networks but determined by the underlying mechanism. Our results imply that the degree distribution is not the only representation of scale-free property, nor the best one.

Results

Definition of Degree–Degree Distance. Given a network $\mathcal{G}(V, E)$, each node $i \in V$ is naturally gifted a scale, the degree k_i (i.e., the number of nodes that are connected to node i). This natural scale is independent of the details of network realization and determined only by network topology, not by extrinsic attributes. In contrast, each link $(i, j) \in E$ is a 2-tuple that has no single comparable scale unless assigned an additional attribute such as a weight or a capacity. The lack of a comparable scale renders links an inferior status in most statistical studies of complex networks.

Our task here is to regain the statistical importance of links by introducing a simple but useful property that is "link oriented"—the degree–degree distance. The definition of degree–degree distance, $\eta(i, j)$, is given by $\log \eta(i, j) = \log \eta(j, i) = |\log k_i - \log k_j|$, $(i, j) \in E$. The degree–degree distance is a natural scale, in the sense that $\eta(i, j)$ is also determined solely by network topology. The degree–degree distance is also dimensionless, in the sense that our definition can be rewritten $\eta(i, j) = \max\{k_i, k_j\}/\min\{k_i, k_j\}$ (i.e., the ratio between the degrees of the nodes at two ends after ordering). Obviously, η lies in the range of $[1, \max\{k_i | i \in V\}]$ in $\mathcal{G}$, which incidentally is the same range in which k lies if $k_{\min} = 1$. Even if it is $\log \eta$ that is the positive measure and should be called a "distance," our work remains focused on η, primarily because we find out later that η plays the same role—if not a better role—as k traditionally does in the study of power laws in complex networks. We will see that the distribution of η, $P(\eta)$, also follows a power law in real life [i.e., $P((1+\epsilon)\eta) \simeq (1+\epsilon)^{-\beta} P(\eta)$], with β the scaling exponent. In reality, any result on η has more statistical significance than k, given that most studied networks are connected and $|E| > |V|$ holds true.

Note that the study of degree–degree relations is not new. That being said, most studies focus on degree–degree correlation features (47, 48) (e.g., assortativity), which only have a statistical sense (*SI Appendix*). In our work, the degree–degree distance is a network topological property that each link possesses per se. It is similar to link weight or link capacity, but it is inherent and thus, more fundamental.

Empirical Result. Whether real-world networks should exhibit power laws has always been worth debating (34, 35). Our aim is to retest the statistical significance of the claim that degree distributions are power law in the real world and most importantly, to look into the statistics of degree–degree distance distributions for comparison. To this end, we collected 32 typical real-world networks that have a wide coverage of economic, biological, informational, social, and technological domains, with their sizes ranging from hundreds to tens of millions of nodes (*SI Appendix*). The networks exhibit composite properties such as being directed or weighted, yet we treat them as simple, undirected graphs so as to investigate their most basic topological structures. Fig. 1 shows the degree distributions and degree–degree distance distributions of 16 characteristic complex networks. The other 16 networks are shown in *SI Appendix*. Fig. 1 shows that in some networks (Fig. 1 *A–D*), both $P(k)$ and $P(\eta)$ appear to be power law, while for others (Fig. 1 *E–P*), $P(k)$ appears to differ from a power law and exhibit complicated turning points before k enters the upper tail.

We further conducted unbiased statistical analysis of full-range fitting to confirm this discovery (*SI Appendix* has details). Although it has been argued that only in a partial range should the fitting be expected to be power law even for a scale-free network, we decided not to add such consideration to our statistical analysis to avoid false positive conclusions (i.e., to avoid such cases that the fitting is partially power law but the network is not scale free). An unbiased, assumption-independent estimation of the appropriate range is known to be a nontrivial issue (49). Fig. 2 shows that only for 16.7% of all 32 networks is a power-law fit of $P(k)$ favored by AICc (the corrected Akaike information criterion). The log-normal fit is the most favored, for 65.6%. Our result is, in general, consistent with ref. 34. In contrast, a power-law fit of $P(\eta)$ is favored for 37.5%, and it is also the most favored. The percentage is more than twice that of $P(k)$. In addition, under the statistical assumption that the accompaniment of an exponential cutoff may not be regarded as a contradiction but a supportive correction to the fitting (35), the percentages of the power laws of $P(k)$ and $P(\eta)$ being favored will increase to 28.1 and 56.2%, respectively. More than half of $P(\eta)$ in real-world networks should still be power-law distributions statistically. Additional tests on the statistical significance of our results are also given in *SI Appendix*.

It is worth noting that there are synthetic or real-world networks in which neither $P(k)$ nor $P(\eta)$ should be power law by apparent reasoning (e.g., the Erdős–Rényi network and the road network). Their nonpower-law distributions are statistically confirmed by AICc comparison (*SI Appendix*), which again suggests that the full-range statistical analysis is unbiased to false positive conclusions. Obviously, some network generation mechanisms never produce a power-law distribution of $P(\eta)$.

Bidirectional Preferential Selection. To explain our findings, we here introduce a heuristic configuration-like model built with no artificial scale set. It consists of the following steps.

1) At the beginning, there are N nodes. Each node i is assigned an importance weight, ω_i, which is randomly sampled from a sample space $\{\omega | \omega = \omega_{\min} + n, n \in \mathbb{N}\}$ by a power-law probability distribution, $\mathrm{Prob}[\omega_i = \omega] = c\omega^{-\alpha}$. Here, c is a normalization constant. Define $\bar{\omega} = N^{-1} \sum_{i=1}^{N} \omega_i$, of which the expectation is $\mathrm{E}[\bar{\omega}] = \sum_{\omega=\omega_{\min}}^{\infty} \omega c\omega^{-\alpha}$.
2) At each time step, two nodes i and j are randomly and independently chosen, and a link is established between them. The probability to choose i and j is $\mathrm{Prob}[i \leftrightarrow j] = (\omega_i / N\bar{\omega})(\omega_j / N\bar{\omega})$. If i and j have been connected before, we discard the link and redo this step without updating the time step.
3) After T time steps, a network of N nodes and T links is generated.

Our bidirectional preferential selection model differs from other models in that it uses a two-way preferentially weighted selection process. A link is more likely to be established when both nodes exhibit a preference for each other. In addition, the number of nodes N is fixed in advance. All network properties are determined by how the T links are distributed (i.e., the establishment of links, not the addition of nodes, determines network properties).

In the continuum limit, the degree distribution of our model is (*Materials and Methods*)

$$P(k) \simeq \int_{\omega_{\min}}^{\infty} c\omega^{-\alpha} \frac{1}{\sqrt{2\pi}\sigma(\omega, T)} \exp\left[-\frac{(k - \mu(\omega, T))^2}{2\sigma^2(\omega, T)}\right] d\omega, \quad [1]$$

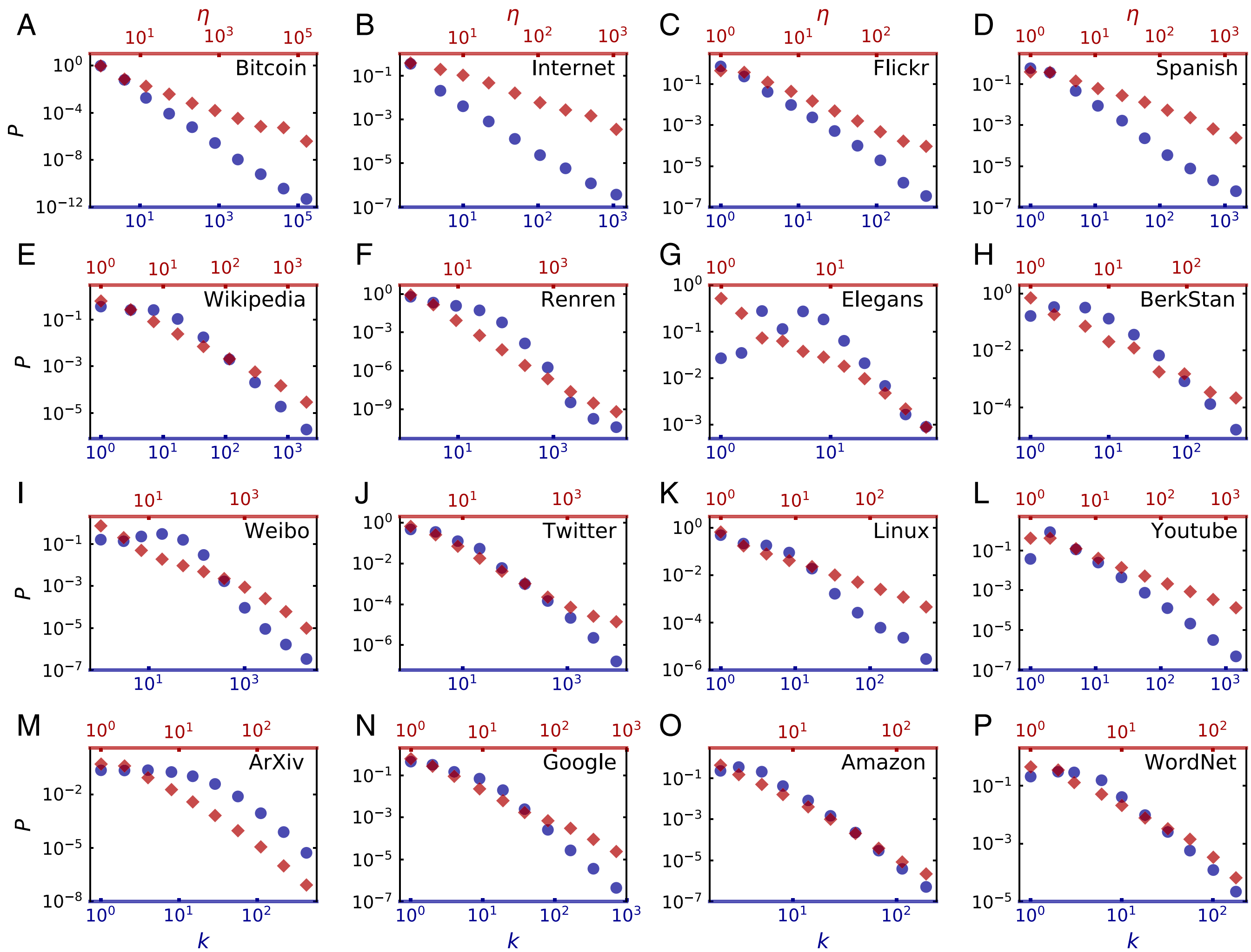

Fig. 1. Degree distributions $P(k)$ (blue) and degree–degree distance distributions $P(\eta)$ (red) of 16 characteristic real-world complex networks (*A*–*P*) of which metadata are given in *SI Appendix*. In general, $P(\eta)$ exhibits a better power law than $P(k)$ exhibits.

where $\mu(\omega, T) = (2\omega/N\bar{\omega})T$ and $\sigma^2(\omega, T) = (2\omega/N\bar{\omega})(1 - 2\omega/N\bar{\omega})T$. Eq. **1** is an integral of two parts: a power-law distribution and a Gaussian packet, the latter of which can be approximated as a Dirac function when $\sigma^2(\omega, T)/\mu^2(\omega, T) \to 0$ as $T, N \to \infty$. We further set $T = N^s/2 = O(N^s)$, $1 < s < 2$, and then, Eq. **1** can be approximated as

$$P(k) \simeq c\left(N^{1-s}\bar{\omega}\right)^{1-\alpha} k^{-\alpha}, \qquad \textbf{[2]}$$

provided that k lies in the upper tail,

$$k \gg \mu(\omega_{\min}, T) = \frac{\omega_{\min}}{\bar{\omega}} N^{s-1}, \qquad \textbf{[3]}$$

which means that k, where the Dirac peak is located, should stay away from the boundary $\omega \approx \omega_{\min}$. The turning point to the upper tail in Eq. **3** is controlled by the parameter s, which quantifies how dense the network is. We see that, given a preferential attachment mechanism, the degree distribution of the generated network is not necessarily power law in the full range but only in the upper tail. Its deviation from being a power law increases as s increases.

Next, after taking the continuum limit, the degree–degree distance distribution is given by (*Materials and Methods*)

$$P(\eta) \simeq \int_{\omega_{\min}}^{\infty}\int_{\omega_{\min}}^{\infty} \frac{c\omega_1^{-\alpha} c\omega_2^{-\alpha}\, d\omega_1\, d\omega_2}{T/\left[N\left(N-1\right)/2\right]} \left(\frac{\eta\mu_2\sigma_1^2 + \mu_1\sigma_2^2}{\eta^2\sigma_1^2 + \sigma_2^2}\right) \cdot \frac{\mu_1\mu_2/4T}{\sqrt{2\pi}\sqrt{\eta^2\sigma_1^2 + \sigma_2^2}} \exp\left[-\frac{\left(\eta\mu_1 - \mu_2\right)^2}{2\left(\eta^2\sigma_1^2 + \sigma_2^2\right)}\right], \qquad \textbf{[4]}$$

where μ_i is $\mu(\omega_i, T)$ and σ_i is $\sigma(\omega_i, T)$, with $i = 1, 2$, respectively. Note that in Eq. **4**, the Dirac peak is determined by $\eta \approx \mu_2/\mu_1$, which receives contributions from not only the boundary $\omega_1 \approx \omega_{\min}$ or $\omega_2 \approx \omega_{\min}$ but also, the neighborhood of the parametric curve determined by $\mu_2/\mu_1 = \text{Const.}$ in the $\{\omega_1, \omega_2\}$ domain. This is true for any η, even when η is close to one. Hence, $P(\eta)$ changes smoothly with η after averaging the $\{\omega_1, \omega_2\}$ domain, and no upper-tail approximation is needed. This explains why $P(\eta)$ exhibits a better power law than $P(k)$. Finally, from Eq. **4**,

$$P(\eta) \simeq \sqrt{\frac{\pi}{8}}\, c^2 N^{\frac{1-s}{2}} \omega_{\min}^{\frac{7}{2}-2\alpha} \bar{\omega}^{-\frac{3}{2}} \eta^{-\alpha+1} \qquad \textbf{[5]}$$

is further derived, indicating that $\beta = \alpha - 1$ in our model.

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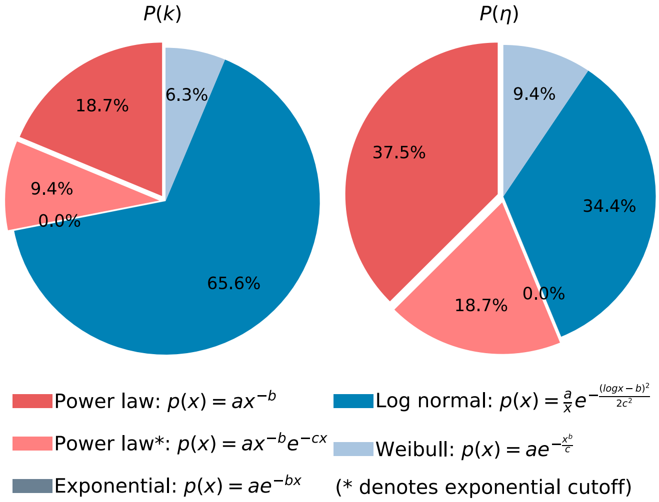

Fig. 2. The best fit of the distributions $P(k)$ and $P(\eta)$ for real-world complex networks determined by statistical analysis. The best fit is the most favored by AICc.

Model Simulation and Validation. As suggested by the bidirectional preferential selection model, $P(\eta)$ should exhibit a smooth power law in the full range of η, yet $P(k)$ should exhibit deviation where k is small. The difference is more obvious when $s \to 2$ (i.e., as the network evolves and becomes more dense). To verify this, we take three real-world evolving complex networks constructed using three regional time-dependent Wikipedia hyperlinks datasets (German, France, and Italy) and examine how they evolve over time. Fig. 3 *A*–*C* show that in roughly a 100-mo period, all three networks have evolved from being relatively sparse into an overly dense stage where the number of links overwhelms the number of nodes. Both $P(k)$ and $P(\eta)$ are approximately power law in the early stage (Fig. 3 *E*–*G*), but in the late stage, all three $P(k)$ exhibit turning points rather than smooth straight lines, while the three $P(\eta)$ differ from $P(k)$ by exhibiting the same power laws—if not stronger (Fig. 3 *I*–*K*). These findings are evidence that whether a network is dense or not determines how better its $P(\eta)$ as a representation of scale-free property is. On the other hand, our simulation (Fig. 3 *D*, *H*, and *L*) matches the real-world networks, evolves in the same way from being sparse to dense, and exhibits a similar comparison between $P(k)$ and $P(\eta)$ in the two stages. Observations on our synthetic networks are further tested by unbiased statistical tests (*SI Appendix*), confirming that $P(\eta)$ is power law, while $P(k)$ may not be. The good match between the three empirical networks and our simulation results indicates that the preferential selection model is more realistic and is better at capturing the complex, unseen mechanism that generates real-world scale-free networks.

Also shown in Fig. 3 are the scaling exponents, α and β, derived from linear fits of $P(k)$ and $P(\eta)$ in the log–log scale, respectively. Note that now α is derived by fitting only the upper tail of $P(k)$ under the a priori assumption that a power-law α "exists" (Eq. **3**). We further apply the same fits on all 32 real-world networks in order to study the relation between α and β. In Fig. 4, the fitting results are plotted, and $\beta \approx 1.0249\alpha - 1.0643$ is confirmed. The relation $\beta = \alpha - 1$ is not limited to our model but is, in fact, more universal. Denote $P(k, \cdot)$ the probability that a randomly chosen link is connected to a node of degree k via one of its ends; then, $P(k, \cdot) \simeq kP(k)$, as there are $P(k)$ fractions of nodes that have degree k and each contributes k links to the total number of links. Suppose $P(\eta) \propto P(k = \eta k_{\min}, \cdot)$, then $P(\eta) \propto \eta^{-\alpha+1}$ given $P(k) \propto k^{-\alpha}$ (*Materials and Methods* has a complete description).

Discussion

Scale-Free Property. We have shown that, for finite-size networks, $P(\eta)$ can statistically be power law when $P(k)$ is not. Thus, we argue that being scale free for complex networks is not a behavior but rather, a property that is determined not by apparent statistics but by intrinsic mechanism. $P(k)$ not being power law does not necessarily indicate that the complex network is not scale free. Regarding the intrinsic mechanism, we can say that the preferential attachment process (36) is scale free because it uses no artificial scale (except $k_{\min}$ or $\omega_{\min}$, which is rather a cutoff introduced to deal with the continuum limit). In our preferential selection model, all results stay unchanged when we double every ω in its mechanism. Thus, the model itself is also scale free, as well as the synthetic networks it generates even if against apparent statistics. For real-world networks, however, the mechanism is often unknown and has to be inferred, which renders the question on scale free extremely difficult.

Note that the word power law has been used in the purely statistical, nonasymptotic sense through the context. Ideally, any distribution given by an infinite-size, scale-free network is trivially asymptotically power law. However, as a matter of fact, being asymptotically power law should not be confused with being statistically power law for finite-size networks. Only regarding the latter can statistically sound conclusions be reached, and only from unbiased, rigorous, and complete statistical tests can the underlying mechanism be correctly inferred and the controversial question on scale free satisfactorily answered.

Conclusion

We have defined the degree–degree distance, $\eta(i, j)$, for each link (i, j) of a complex network, given by $\log \eta(i, j) = |\log k_i - \log k_j|$, as a fundamental network topological property. We find that in many real-world networks the distribution of degree–degree distance $P(\eta)$ also follows a power law, which is, surprisingly, more statistically significant than the power law of the distribution of degree $P(k)$. Only 28.1% of the considered networks have degree distributions that can be properly modeled by power laws, yet 56.2% have degree–degree distance distributions that can be modeled by power laws. We explain our findings by introducing a configuration-like network generation mechanism called bidirectional preferential selection. Our model successfully describes the power laws of $P(k)$ and $P(\eta)$ and also predicts the deviation of $P(k)$ from being power law when the network evolves to an overly dense stage, which is further verified and confirmed by analysis on the evolution of real-world complex networks. The model also predicts a universal relation between α and β, the scaling exponents of $P(k)$ and $P(\eta)$, respectively, that $\beta = \alpha - 1$. Once again, the relation is justified by real-world data analysis.

Our results allow us to address doubts about the abundance of scale-free networks by proposing that being scale free is a property determined not by apparent statistics but by an underlying mechanism. Preferential attachment, for example, is considered scale free, even when it may generate some finite-size networks that do not have statistically significant power-law $P(k)$. We conclude that $P(\eta)$ is a better representation of the scale-free property of a complex network, especially when the network is dense. In future research, we look forward to a complete statistical test on $P(\eta)$ to consolidate our conclusion and further investigations into the deeper mechanisms responsible for the scale-free property.

Materials and Methods

Derive Degree–Degree Distance Distribution from Joint Probability Distribution. The joint probability distribution $P(x, y)$ is defined as the fraction of links that are equal to the 2-tuple (x, y) (i.e., the fraction of links that

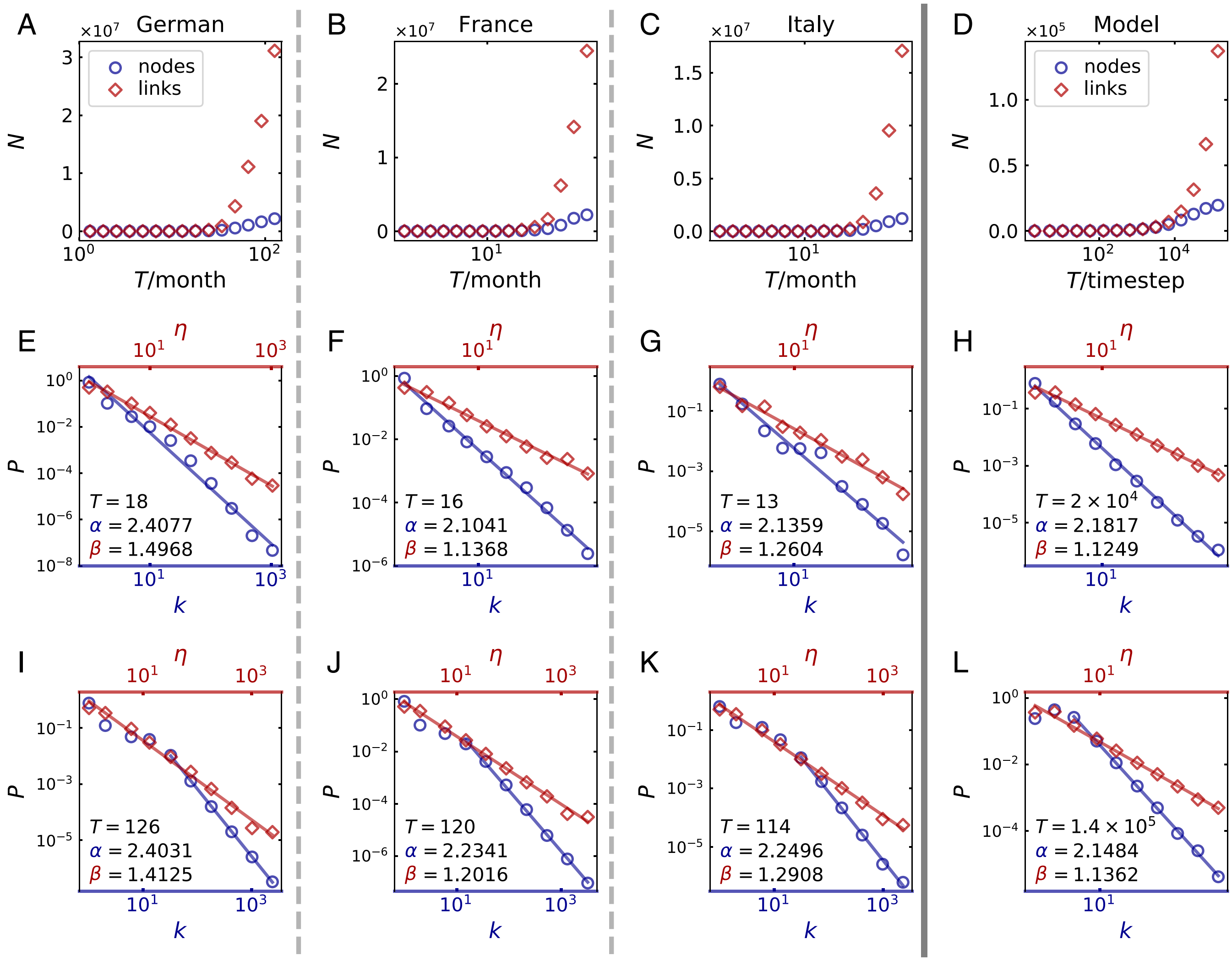

Fig. 3. Three real-world evolving complex networks, which are constructed from regional, time-dependent Wikipedia hyperlinks, as well as simulation result of the bidirectional preferential selection model for comparison. *A* shows how the numbers of nodes (blue circles) and links (red diamonds) have evolved in 126 mo in the Wikipedia (German) hyperlinks network. *E* shows the temporal degree distribution $P(k)$ (blue circles) and degree–degree distance distribution $P(\eta)$ (red diamonds) of the Wikipedia (German) hyperlinks network at an early stage (in the 18th month). *I* shows the same $P(k)$ and $P(\eta)$ but at a later stage (in the 126th month). Linear fits (of the upper tails if necessary) of $P(k)$ and $P(\eta)$ are also given (solid lines). Similarly, *B*, *F*, and *J* are from Wikipedia (France). *C*, *G*, and *K* are from Wikipedia (Italy). On the other hand, *D*, *H*, and *L* are our simulation result from the bidirectional preferential selection model. The parameters are chosen as $N = 2 \times 10^4$, $\omega_{\min} = 1$, and $\alpha = 2$.

connect nodes of degree x and degree y). It is obvious that $P(x, y) = P(y, x)$ by definition. The cumulative probability distribution of η is

$$P(H \le \eta) = P(Y/\eta < X \le Y\eta) = \int_{\max\{y/\eta, k_{\min}\}}^{\eta y} \int_{k_{\min}}^{\infty} P(x, y)dxdy,$$

where $\ln H = |\ln X - \ln Y|$. Thus, the probability density distribution of η is

$$\begin{aligned} P(\eta) &= P(\eta < H \le \eta + d\eta) = dP(H \le \eta)/d\eta \\ &= \int_{k_{\min}}^{\infty} P(\eta y, y)ydy + \int_{\eta k_{\min}}^{\infty} P(y/\eta, y)\frac{y}{\eta^2}dy \\ &= \int_{k_{\min}}^{\infty} 2P(\eta y, y)ydy. \end{aligned} \quad [6]$$

If the joint probability distribution can be factorized into a product of two power laws, $P(k_1, k_2) \simeq c k_1^{-\alpha+1} k_2^{-\alpha+1}$, then Eq. **6** immediately yields $P(\eta) \simeq c(\alpha-2)^{-1} k_{\min}^{-2\alpha+4} \eta^{-\alpha+1}$ and so, $\beta = \alpha - 1$. Another example is the Barabási–Albert model with the degree distribution $P(k) = 2k_{\min}(k_{\min}+1)/(k(k+1)(k+2))$ that simply implies $\alpha = 3$. Meanwhile, its joint probability distribution

$$P(k_1, k_2) = \frac{2k_{\min}(k_{\min}+1)}{k_1(k_1+1)k_2(k_2+1)}\left[1 - \frac{\binom{2k_{\min}+2}{k_{\min}+1}\binom{k_1+k_2-k_{\min}}{k_2-k_{\min}}}{\binom{k_1+k_2+2}{k_2+1}}\right] \quad [7]$$

is given by ref. 50. Eq. **6** together with Eq. **7** yields

$$P(\eta) \simeq 4(k_{\min}+1)\left(1 + k_{\min}\ln\frac{k_{\min}}{1+k_{\min}}\right)\eta^{-2} + \cdots,$$

confirming that $\beta = 2$. Note that Eq. **7** cannot be factorized but is a mixed distribution with nonzero correlation (2), yet $\beta = \alpha - 1$ is still satisfied.

Probability Distributions of the Bidirectional Preferential Selection Model. Suppose a node is initially assigned an importance weight ω_i. The probability that it has degree k after T time steps is a binomial distribution, $\binom{T}{k}p(\omega_i)^k(1-p(\omega_i))^{1-k}$, where $p(\omega_i) \simeq 2\omega_i/N\bar{\omega}$ is the probability that the node is chosen in one time step (given $T \ll N^2$). Because the probability

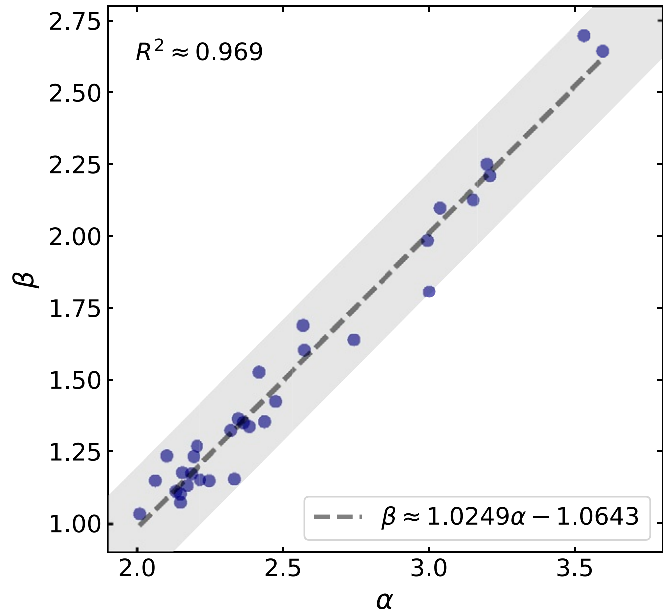

Fig. 4. Relation between the two scaling exponents, α and β, of degree distribution and degree–degree distance distribution, respectively. The fitting function is $\beta \approx 1.0249\alpha - 1.0643$, and $R^2 \approx 0.969$. The shaded area is within one SE.

that the node is initially assigned $\omega_i = \omega$ is $c\omega^{-\alpha}$, taking the sum of all possibilities yields

$$P(k) \simeq \sum_{\omega=\omega_{\min}}^{\infty} c\omega^{-\alpha} \binom{T}{k} \left(\frac{2\omega}{N\bar{\omega}}\right)^{k} \left(1 - \frac{2\omega}{N\bar{\omega}}\right)^{T-k}. \qquad [8]$$

Note that the binomial distribution (Eq. **8**) can be approximated by a continuous Gaussian distribution when $(2\omega/N\bar{\omega})T = O(N^{s-1})$ is large enough [given $T = O(N^s)$], which gives rise to Eq. **1**.

We now calculate the joint probability distribution $P(k_1, k_2)$. The conditional probability that two nodes of importance weights ω_i and ω_j are connected in T time steps is Prob $[i \leftrightarrow j | \{\omega_i, \omega_j\}] = 1 - (1 - (\omega_i/N\bar{\omega})(\omega_j/N\bar{\omega}))^T \simeq (\omega_i/N\bar{\omega})(\omega_j/N\bar{\omega})T$. Therefore,

$$\text{Prob}\left[\{\omega_i, \omega_j\} | i \leftrightarrow j\right] = \frac{\text{Prob}\left[i \leftrightarrow j | \{\omega_i, \omega_j\}\right] \cdot \text{Prob}\left[\{\omega_i, \omega_j\}\right]}{\text{Prob}\left[i \leftrightarrow j\right]}$$

is obtained using Bayes' rule. So, the probability of choosing a link (i, j) when the two nodes have importance weights ω_i and ω_j is

$$\text{Prob}\left[\{\omega_i, \omega_j\} | i \leftrightarrow j\right] = \frac{(\omega_i/N\bar{\omega})(\omega_j/N\bar{\omega})T \cdot c\omega_i^{-\alpha} c\omega_j^{-\alpha}}{T/[N(N-1)/2]}. \qquad [9]$$

Note that Eq. **9** can be factorized into Prob $\left[\{\omega_i, \omega_j\} | i \leftrightarrow j\right] = f(\omega_i) f(\omega_j)$. As in Eq. **8**, taking the sum of all possibilities yields

$$P(k_1, k_2) \simeq \left[\sum_{\omega=\omega_{\min}}^{\infty} f(\omega) \binom{T}{k_1} \left(\frac{2\omega}{N\bar{\omega}}\right)^{k_1} \left(1 - \frac{2\omega}{N\bar{\omega}}\right)^{T-k_1}\right] \cdot \left[\sum_{\omega=\omega_{\min}}^{\infty} f(\omega) \binom{T}{k_2} \left(\frac{2\omega}{N\bar{\omega}}\right)^{k_2} \left(1 - \frac{2\omega}{N\bar{\omega}}\right)^{T-k_2}\right]. \qquad [10]$$

Finally, taking the Gaussian approximation of Eq. **10** and putting it into Eq. **6** produces the final form (Eq. **4**).

Data Availability. All data are publicly available at the Colorado Index of Complex Networks at https://icon.colorado.edu.

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